**Complex number 2**

**1.** Given $z\_{1}=2+i$ , $z\_{2}=3-4i$ and $\frac{1}{z\_{3}}=\frac{1}{z\_{1}}+2z\_{2}$ , find $z\_{3}$.

 Write your answer in the standard form of $a+bi$.

 Hence, find the modulus and principal argument of $z\_{3}$.

 State your answers correct to three decimal places.

 $\frac{1}{z\_{3}}=\frac{1}{z\_{1}}+2z\_{2}⟹z\_{3}=\frac{z\_{1}}{1+2z\_{1}z\_{2}}=\frac{2+i}{1+2\left(2+i\right)\left(3-4i\right)}=\frac{2+i}{21-10i}=\frac{\left(2+i\right)\left(21+10i\right)}{\left(21-10i\right)\left(21+10i\right)}=\frac{32+41i}{541}=\frac{32}{541}+\frac{41}{541}i$

 $\left|z\_{3}\right|=\sqrt{\left(\frac{32}{541}\right)^{2}+\left(\frac{41}{541}\right)^{2}}=\frac{\sqrt{2705}}{541}≈0.0961360711567≈0.096$(to 3 dec.places)

 $Arg\left(z\_{3}\right)=tan^{-1}\frac{\frac{41}{541}}{\frac{32}{541}}=tan^{-1}\frac{41}{32}≈0.9080668189019≈0.908 radians$(to 3 dec.places)

**2.** Given that $\left|z-1\right|=2\left|z+1\right|$, find the Cartesian equation of the locus of the point P representing complex number z. .

 Hence, sketch the locus of the point P on an Argand diagram.



 Let $z=x+yi$

 $\left|z-1\right|=2\left|z+1\right|$

 $⟹\left|x+yi-1\right|=2\left|x+yi+1\right|$

 $⟹\left|\left(x-1\right)+yi\right|^{2}=4\left|\left(x+1\right)+yi\right|^{2}$

 $⟹\left(x-1\right)^{2}+y^{2}=4\left[\left(x+1\right)^{2}+y^{2}\right]$

 $⟹3x^{2}+3y^{2}+10 x+3=0$

 $⟹x^{2}+y^{2}+\frac{10}{3} x+1=0$

 Locus is a circle centre = $-\frac{5}{3}$ ,

 radius = $\sqrt{\left(-\frac{5}{3}\right)^{2}-1}=\frac{4}{3}$

**3.** Solve the equation $z^{5}+32i=0$.

 $z^{5}=-32i=32\left(\cos(\frac{π}{2})+i\sin(\frac{π}{2})\right)=32 cis \left(2kπ+\frac{π}{2}\right) , where kϵZ.$

By de Moirves’ Theorem,

 $z\_{k}=\left(-32i\right)^{\frac{1}{5}}=\left[32 cis \left(2kπ+\frac{π}{2}\right)\right]^{\frac{1}{5}}=2 cis \left(\frac{2kπ+\frac{π}{2}}{5}\right), where k=0,1,2,3,4.$

 $z\_{0}=2 cis \left(\frac{π}{10}\right)≈1.9021-0.6180i$

 $z\_{1}=2 cis \left(\frac{3π}{10}\right)≈1.1756+1.6180i$

 $z\_{2}=2 cis \left(\frac{5π}{10}\right)=-2i$

 $z\_{3}=2 cis \left(\frac{7π}{10}\right)≈-1.1756+1.6180i$

 $z\_{4}=2 cis \left(\frac{9π}{10}\right)≈-1.9021-0.6180i$

**4.** If the equation $z^{3}+az+b=0$ has a root $z=-1+i$ where $a,b$ are real numbers, find the values of $a,b$. Show that $z=-1-i$ is also a root of the equation.

 $\left(-1+i\right)^{3}+a\left(-1+i\right)+b=0⟹\left(2+2i\right)+a\left(-1+i\right)+b=0⟹\left(2-a+b\right)i+\left(2+a\right)=0$

 $\left\{\begin{array}{c}2-a+b=0\\2+a=0\end{array}\right.⟹\left\{\begin{array}{c}a=-2\\b=-4\end{array}\right.$

 The equation becomes $z^{3}-2z-4=0$

 Since $\left(-1-i\right)^{3}-2\left(-1-i\right)-4=\left(2-2i\right)-2\left(-1-i\right)-4=0, z=-1-i$ is also a root of the equation.

**5.** **(a)** One of the roots of the equation $4x^{3}+x+5=0$ is an integer. Find this root and write down a quadratic equation for the remaining roots. Find these roots, expressing your answer in the satandard form of $a+bi$.

 **(b)** By writing $y=\frac{1}{x}$, find the roots of the equation $5y^{3}+y^{2}+4=0$,

 giving the complex roots in the form $a+bi$.

 **(a)** $f\left(x\right)=4x^{3}+x+5, f\left(-1\right)=4\left(-1\right)+\left(-1\right)+5=0⟹\left(x+1\right)$ is a factor of $f\left(x\right)$.

 By division, we have $4x^{3}+x+5=\left(x+1\right) \left(4 x^{2}-4 x+5\right)=0$

 $∴x=-1 or \frac{1}{2}-i or \frac{1}{2}+i$.

 **(b)** $5y^{3}+y^{2}+4=0⟹5\left(\frac{1}{x}\right)^{3}+\left(\frac{1}{x}\right)^{2}+4=0⟹4x^{3}+x+5=0$

 $∴\frac{1}{y}=-1 or \frac{1}{y}=\frac{1}{2}-i or \frac{1}{y}=\frac{1}{2}+i$

 $∴y=-1 or \frac{2}{5}+\frac{4}{5}i or \frac{2}{5}-\frac{4}{5}i$

**6.** Find the roots of the equation $\left(z-iα\right)^{3}=i^{3}$ , where $α$ is a real constant.

 **(a)** Show that the points representing the roots of the above equation form an equilateral triangle.

 **(b)** Solve the equation $\left[z-\left(1+i\right)\right]^{3}=\left(2i\right)^{3}$.

 **(c)** If $ω$ is a root of the equation $ax^{2}+bx+c=0$, where $a,b,c\in R$ and $a\ne 0$ , show that the conjugate $ω'$ is also a root of this equation.

 **(d)** Hence, or otherwise, obtain a polynomial equation of degree six with three of its roots also the roots of the equation $\left(z-1\right)^{3}=i^{3}$

 **(a)** $\left(z-iα\right)^{3}=i^{3}=-i=cis \left(2kπ+\frac{3π}{2}\right), k\in R$

 $z-iα=\left[cis \left(2kπ+\frac{3π}{2}\right)\right]^{1/3}=cis \left(\frac{2kπ+\frac{3π}{2}}{3}\right), k=0,1,2$.

 $∴z=iα+cis \left(\frac{2kπ+\frac{3π}{2}}{3}\right), k=0,1,2$.

 $z\_{0}=iα+cos \left(\frac{0+\frac{3π}{2}}{3}\right)+i sin \left(\frac{0+\frac{3π}{2}}{3}\right)=\left(α+1\right)i$

 $z\_{1}=iα+cos \left(\frac{2π+\frac{3π}{2}}{3}\right)+i sin \left(\frac{2π+\frac{3π}{2}}{3}\right)=-\frac{\sqrt{3}}{2}+\left(α-\frac{1}{2}\right)i$

 $z\_{2}=iα+cos \left(\frac{4π+\frac{3π}{2}}{3}\right)+i sin \left(\frac{4π+\frac{3π}{2}}{3}\right)=\frac{\sqrt{3}}{2}+\left(α-\frac{1}{2}\right)i$

 $\left|z\_{0}-z\_{1}\right|=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^{2}+\left[\left(α+1\right)-\left(α-\frac{1}{2}\right)\right]^{2}}=3$

 $\left|z\_{1}-z\_{2}\right|=\sqrt{\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)^{2}+\left[\left(α-\frac{1}{2}\right)-\left(α-\frac{1}{2}\right)\right]^{2}}=3$

 $\left|z\_{2}-z\_{0}\right|=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^{2}+\left[\left(α-\frac{1}{2}\right)-\left(α+1\right)\right]^{2}}=3$

 Thus the points representing the roots of the above equation form an equilateral triangle.

 **(b)** $\left[z-\left(1+i\right)\right]^{3}=\left(2i\right)^{3}⟹\left[\frac{z-\left(1+i\right)}{2}\right]^{3}=i^{3}⟹\left[\frac{z-1}{2}-\frac{1}{2}i\right]^{3}=i^{3}$

 By (a), $\frac{z\_{0}-1}{2}=\left(\frac{1}{2}+1\right)i⟹z\_{0}=1+3i$

 $\frac{z\_{1}-1}{2}=-\frac{\sqrt{3}}{2}+\left(\frac{1}{2}-\frac{1}{2}\right)i⟹z\_{1}=1-\sqrt{3}$

 $\frac{z\_{2}-1}{2}=\frac{\sqrt{3}}{2}+\left(\frac{1}{2}-\frac{1}{2}\right)i⟹z\_{2}=1+\sqrt{3}$

 **(c)** If $ω$ is a root of the equation $ax^{2}+bx+c=0$, then $aω^{2}+bω+c=0$

$\overbar{aω^{2}+bω+c}=\overbar{0}⟹\overbar{a}\overbar{ω^{2}}+\overbar{b}\overbar{ω}+\overbar{c}=0⟹a\overbar{ω}^{2}+b\overbar{ω}+c=0$ , since $a,b,c\in R$

 $ω^{'}=\overbar{ω}$ is also a root of this equation.

 **Alternatively**, we can set $ω=u+vi, ω^{'}=u-vi$

$$aω^{2}+bω+c=0⟹a\left(u+vi\right)^{2}+b\left(u+vi\right)+c=0$$

 $⟹\left(a u^{2}-a v^{2}+bu+c\right)+\left(2auv+bv\right)i=0$

 $⟹\left(a u^{2}-a v^{2}+bu+c=0\right)and \left(2auv+bv\right)=0$

 $aω'^{2}+bω^{'}+c=a\left(u-vi\right)^{2}+b\left(u-vi\right)+c=\left(a u^{2}-a v^{2}+bu+c\right)-\left(2auv+bv\right)i=0$

 Thus, $ω^{'}$ is also a root of the equation.

 **(d) Method 1 (More complicate, but it satisfies the former part of quadratics)**

 Replace $iα$ by 1 in part (a), the roots of $\left(z-1\right)^{3}=i^{3}$ are

 $z\_{0}=1+i$ **,** $z\_{1}=1-\frac{\sqrt{3}}{2}-\frac{1}{2}i$ **,** $z\_{2}=1+\frac{\sqrt{3}}{2}-\frac{1}{2}i$

 Their conjugates are $z\_{0}^{'}=1-i$ , $z\_{1}^{'}=1-\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $z\_{2}^{'}=1+\frac{\sqrt{3}}{2}+\frac{1}{2}i$

 Hence, the required polynomial equation of degree six is

$\left[z-\left(1+i\right)\right]\left[z-\left(1-i\right)\right]\left[z-\left(1-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\right]\left[z-\left(1-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right]\left[z-\left(1+\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\right]\left[z-\left(1+\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right]=0$ ,

 which is the product of three quadratics:

$$\left(z^{2}-2z+2\right)\left(z^{2}+\sqrt{3}z-2z-\sqrt{3}+2\right)\left(z^{2}-\sqrt{3}z-2z+\sqrt{3}+2\right)=0$$

 $\left(z^{2}-2z+2\right)\left(z^{4}-4z^{3}+5 z^{2}-2z+1\right)=0$

$z^{6}-6 z^{5}+15 z^{4}-20 z^{3}+15 z^{2}-6 z+2=0$

 **Method 2 (faster, but it does not use the former part of quadratics)**

 However, we you simply find a polynomial equation of degree six with real coefficients.

 Consider $\left[\left(z-1\right)^{3}-i^{3}\right]\left[\left(z-1\right)^{3}+i^{3}\right]=0$,

 which obviously has the roots of the equation $\left(z-1\right)^{3}=i^{3}$.

 $\left[\left(z-1\right)^{3}+i\right]\left[\left(z-1\right)^{3}-i\right]=0$ (note that the conjugates are also roots)
 $\left(z-1\right)^{6}-i^{2}=0$

 $\left(z-1\right)^{6}+1=0$

 $z^{6}-6 z^{5}+15 z^{4}-20 z^{3}+15 z^{2}-6 z+2=0$

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